MTH 422
Exam 1
Spring 2024

100 points possible. 10 problems at 10 points each.

1. Take-home problem due Tuesday. Upload to Blackboard.

2. A production facility contains two machines that are used to rework items that are initially defective. Let X be the number of hours that the first machine is in use, and let Y be the number of hours that the second machine is in use, on a randomly chosen day. Assume that X and Y have joint probability density function given by

$$f(x,y) = \begin{cases} \frac{3}{32}(x^2+y^2) & \text{if } (x,y) \in [0,2] \times [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both machines are in operation for more than one hour?

3. Each week, a subcommittee of five individuals is formed from among the members of a committee comprising eight individuals. Two subcommittee members are then assigned to lead the subcommittee, one as chair and the other as secretary.

Calculate the maximum number of consecutive weeks that can elapse without having the subcommittee contain five individuals who have previously served together with the same subcommittee chair.

4. On Main Street, a driver's speed just before an accident is uniformly distributed on $[10, 40]$. Given the speed, the resulting loss from the accident is exponentially distributed with mean equal to four times the speed.

Calculate the variance of a loss due to an accident on Main Street.

5. Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which

$$E(X) = 6, \quad E(X^2) = 38.6, \quad E(Y) = 8, \quad E(Y^2) = 70.6, \quad \text{Var}(X + Y) = 15$$

Let $C_1 = X + Y$ denote the size of the combined claims before the application of a 10% surcharge on the hospital portion of the claim, and let $C_2 = X + 1.10Y$ denote the size of the combined claims after the application of that surcharge.

Calculate $\text{Cov}(C_1, C_2)$.

6. Let $X_1, \ldots, X_8$, and X_9 be independent normally distributed random variables, each with mean 4 and standard deviation 5.

Which of the following has a chi-square distribution?

(A) $\sum_{i=1}^{9} X_i^2$ (B) $\frac{1}{9}\sum_{i=1}^{9} X_i^2$ (C) $\frac{1}{5}\sum_{i=1}^{9} (X_i - 4)^2$

(D) $\frac{1}{9}\sum_{i=1}^{9} (X_i - 4)^2$ (E) $\frac{1}{25}\sum_{i=1}^{9} (X_i - 4)^2$

7. Let the distribution of W be $F(20, 5)$. Find the following.

(a) $F_{0.01}(20, 5)$

(b) $F_{0.99}(20, 5)$

(c) $P(0.304 \leq W \leq 4.56)$

8. Claim amounts at an insurance company are independent of one another. In year one, claim amounts are modeled by a normal random variable X with mean 100 and standard deviation 20. In year two, claim amounts are modeled by the random variable $Y = 1.05X + 10$.

Calculate the probability that a random sample of 25 claim amounts in year two average between 110 and 120.

9. Let $\overline{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval $(0, 2)$. Approximate $P(1.0 \leq \overline{X} \leq 1.3)$.

10. If X is a random variable with mean 32 and variance 25, use Chebyshev's inequality to find

(a) a lower bound for $P(20 < X < 44)$;

(b) an upper bound for $P(|X - 32| \geq 15)$.