MTH 361
Exam 2
Spring 2025

100 points possible. 9 problems at 11 points each, plus 1 free point.

Notations:

$\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$	positive elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
$\mathbb{Z}^*$, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$	nonzero elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

1. Find the order of the given element of the direct product.

$$(3, 10, 9) \text{ in } \mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$$

2. What is the largest order among the orders of all the cyclic subgroups of the given group.

(a) $\mathbb{Z}_{14} \times \mathbb{Z}_5 \times \mathbb{Z}_{18}$

(b) $\mathbb{Z}_8 \times \mathbb{Z}_{14} \times \mathbb{Z}_{10}$

3. Disregarding the order of the factors, write direct products of two or more groups of the form $\mathbb{Z}_n$ so that the resulting product is isomorphic to $\mathbb{Z}_{180}$ in as many ways as possible.

4. Find all abelian groups, up to isomorphism, of the given order. Write them in both forms (1) and (2) from the Fundamental Theorem.

As a reminder, in form (1) every subscript is a power of a prime, and in form (2) every subscript divides the next subscript.

Order 1782

5. (a) Find all cosets of the subgroup $5\mathbb{Z}$ of $\mathbb{Z}$. How many distinct cosets did you find?

(b) Find all cosets of the subgroup $\langle 3 \rangle$ of $\mathbb{Z}_{18}$. How many distinct cosets did you find?

6. Find all left cosets of the subgroup $H = \{E, D\}$ of the group G given by this table.

	E	A	B	C	D	F	G	H
E	E	A	B	C	D	F	G	H
A	A	E	C	B	F	D	H	G
B	B	H	D	A	G	C	E	F
C	C	G	F	E	H	B	A	D
D	D	F	G	H	E	A	B	C
F	F	D	H	G	A	E	C	B
G	G	C	E	F	B	H	D	A
H	H	B	A	D	C	G	F	E

(b) Rewrite the above table in the order exhibited by the left cosets in part (a). Do you seem to get a coset group of order 4? If so, is it isomorphic to $\mathbb{Z}_4$ or to the Klein 4-group V?

7. Prove that the given map ϕ is an isomorphism of the first group with the second.

$$\langle \mathbb{R}, + \rangle \text{ with } \langle \mathbb{R}^+, \cdot \rangle \text{ where } \phi(x) = 4^x \text{ for } x \in \mathbb{R}.$$

8. Let t and u be fixed positive integers. Let

$$H = \{jt + ku : j, k \in \mathbb{Z}\}.$$

Prove that H is a subgroup of $\langle \mathbb{Z}, + \rangle$.

9. Mark each of the following true or false. (No reasons required.)

(a) Every abelian group of prime order is cyclic.

(b) Every abelian group of order divisible by 6 contains a cyclic subgroup of order 6.

(c) Groups of finite order must be used to form an external direct product.

(d) Every element in $\mathbb{Z}_4 \times \mathbb{Z}_8$ has order 8.

(e) $\mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to S_8.

(f) The order of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.