MTH 361
Exam 1
Spring 2025

100 points possible. 9 problems at 11 points each, plus 1 free point.

Notations:

$\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$	positive elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
$\mathbb{Z}^*$, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$	nonzero elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

1. Determine whether the binary operation $*$ defined is commutative and whether $*$ is associative. Justify.

$$* \text{ defined on } \mathbb{Z}^+ \text{ by letting } a * b = b^a$$

2. Determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ from the definition of a group that does not hold. (As a reminder, $\mathcal{G}_1$ is associativity, $\mathcal{G}_2$ is the existence of an identity, and $\mathcal{G}_3$ is the existence of inverses.)

$$\text{Let } * \text{ be defined on } \mathbb{R} \text{ by letting } a * b = ab$$

3. Which of the following groups are cyclic? For each cyclic group, list all the generators of the group.

$G_1 = \langle \mathbb{Z}, + \rangle$
$G_2 = \langle \mathbb{R}^+, \cdot \rangle$
$G_3 = \{3^n : n \in \mathbb{Z}\}$ under multiplication
$G_4 = \langle 5\mathbb{Z}, + \rangle$ under addition
$G_5 = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ under addition

4. Compute the indicated product involving the following permutations in S_6.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 4 & 6 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 4 & 1 & 2 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 6 & 3 & 5 \end{pmatrix}$$

(a) $\tau^2\sigma$

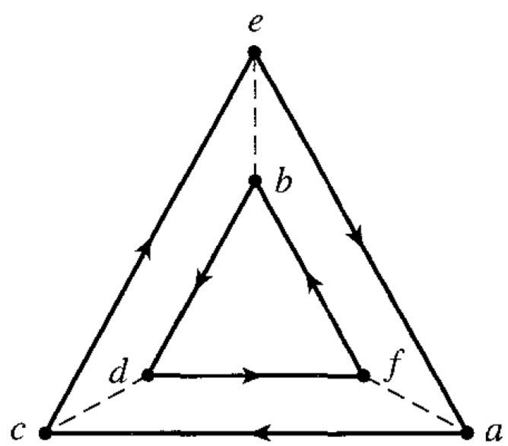
(b) $\mu\sigma^2$

5. Express the permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles, and then as a product of transpositions.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 8 & 3 & 7 & 1 & 6 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 7 & 6 & 1 & 4 & 5 & 2 \end{pmatrix}$

6. The given digraph represents a group. Take e as the identity. Compute the indicated products. Make sure your letters are legible.



(a) $(a^2b)a^3$

(b) $(ab)(a^3b)$

(c) $b(a^2b)$

7. Find all subgroups of the group $\mathbb{Z}_{24}$, and draw the subgroup diagram for the subgroups.

8. Let A be an infinite set. Let H be the set of all $\sigma \in S_A$ such that the number of elements moved by σ is finite.

(Recall that for $\sigma \in S_A$ and $a \in A$, we say σ *moves* a if $\sigma(a) \neq a$.)

Prove that H is a subgroup of S_A.

9. Mark each of the following true or false. (No reasons required.)

(a) A binary operation $*$ on a set S is commutative if there exist $a, b \in S$ such that $a * b = b * a$.

(b) An equation of the form $a * x * b = c$ always has a unique solution in a group.

(c) A cyclic group has a unique generator.

(d) Every element of a group generates a cyclic subgroup of the group.

(e) Every permutation is a cycle.

(f) Every group of order ≤ 4 is cyclic.