**Math 361**
**Exam 2**

100 points possible.

**1.** (20 pts.)

(a) Express (1 2 3 5 7)(2 4 7 6) as a product of disjoint cycles.

(b) Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 3 & 6 \end{pmatrix}$ as a product of transpositions.

**2.** (20 pts.) Let $S_3$ denote the group of all permutations of $\{1, 2, 3\}$. In cycle notation,

$$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}.$$

Let $H = \langle (1\ 2) \rangle$.

(a) Find all left cosets of $H$ in $S_3$.

(b) Find all right cosets of $H$ in $S_3$.

(c) Is $H$ normal in $S_3$?

**3.** (20 pts.) Let $(G, \bullet)$, $(H, *)$, and $(K, \diamond)$ be groups. Assume $\theta : G \to H$ and $\phi : H \to K$ are group homomorphisms. Prove that the composite map $\phi\theta : G \to K$ is a group homomorphism. In your proof, do not omit the operation symbols $\bullet$, $*$, and/or $\diamond$. Include them explicitly.

**4.** (20 pts.) Let $G$ be a group and let $W = G \times G$ be the direct product of $G$ with itself. That is, $W = \{(x, y) \mid x, y \in G\}$ with operation $(x, y)(z, w) = (xz, yw)$. Define $\phi : G \to W$ by $\phi(x) = (x, x)$ for all $x \in G$.

(a) Prove that $\phi$ is a homomorphism.

(b) Prove that $\phi$ is one-to-one.

(c) By a theorem, we know that the image of $\phi$, i.e., the set $\phi(G) = \{(x, x) \mid x \in G\}$, must be a subgroup of $W$. Assume that $G$ is abelian. Prove that $\phi(G)$ must be normal in $W$.

(d) Now prove the converse of part (c). That is, assume that $\phi(G)$ is normal in $W$. Prove that $G$ must be abelian.

**5.** (20 pts.) Let $G$ be a group of order $n$ ($n > 1$). Prove that $G$ has a proper nontrivial subgroup if and only if $n$ is not a prime.