Math 361
Exam 1

100 points possible.

1. (20 pts.) Determine whether the binary operator $*$ defined is commutative and whether $*$ is associative.

$$* \text{ defined on } \mathbb{Z} \text{ by } a * b = 2a + 2b$$

2. (20 pts.) Let G be a group with identity e, and let $a, b \in G$. Prove that if $ab = e$, then $ba = e$.

3. (20 pts.) Let K be a subgroup of a group G. For $a, b \in G$, define $a \sim b$ to mean $ab^{-1} \in K$. Prove the following.

(a) For all $a \in G$, $a \sim a$.

(b) For all $a, b \in G$, if $a \sim b$, then $b \sim a$.

(c) For all $a, b, c \in G$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

4. (20 pts.) Let G be a group. Suppose that every element of G (except for the identity) generates a cyclic subgroup of order 2. Prove G is abelian.

5. (20 pts.) Let A be a set and let $\sigma \in S_A$. For a fixed $a \in A$, the set

$$\mathcal{O}_{a,\sigma} = \{\sigma^n(a) \mid n \in \mathbb{Z}\}$$

is called the *orbit of* a *under* σ. Find the orbit of 1 under the following permutation in S_6.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 3 & 6 \end{pmatrix}$$

Optional Bonus. (Make as much progress as you can.) Let G be a set with an associative binary operation and an element e such that for all $x \in G$,

(1) $ex = x$, and

(2) there exists $x' \in G$ such that $x'x = e$.

Prove that G is a group. That is, prove that we also have $xe = x$ and $xx' = e$.