MTH 361
Exam 1
Spring 2024

100 points possible. 9 problems at 11 points each, plus 1 free point.

Notations:

$\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$	positive elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
$\mathbb{Z}^*$, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$	nonzero elements of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

1. Determine whether the binary operation $*$ defined is commutative and whether $*$ is associative. Justify.

$$* \text{ defined on } \mathbb{Z} \text{ by letting } a * b = ab + 3$$

2. Determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ from the definition of a group that does not hold. (As a reminder, $\mathcal{G}_1$ is associativity, $\mathcal{G}_2$ is the existence of an identity, and $\mathcal{G}_3$ is the existence of inverses.)

$$\text{Let } * \text{ be defined on } \mathbb{R}^+ \text{ by letting } a * b = a$$

3. Nine groups are given below. Give a complete list of all subgroups relations, of the form $G_i \leq G_j$, that exist between these given groups G_1, G_2, ... , G_9.

Note: If, for example, you determine that $A \leq B$ and $B \leq C$ and $C \leq D$, then you may abbreviate these by writing $A \leq B \leq C \leq D$, instead of writing out *all* of the relations $A \leq A$, $A \leq B$, $A \leq C$, $A \leq D$, $B \leq B$, $B \leq C$, $B \leq D$, $C \leq C$, $C \leq D$, and $D \leq D$.

$G_1 = \mathbb{Z}$ under addition
$G_2 = \mathbb{R}$ under addition
$G_3 = \{4^n \mid n \in \mathbb{Z}\}$ under multiplication
$G_4 = 8\mathbb{Z}$ under addition
$G_5 = \mathbb{Q}^+$ under multiplication
$G_6 = 16\mathbb{Z}$ under addition
$G_7 = \{\pi^n \mid n \in \mathbb{Z}\}$ under multiplication
$G_8 =$ the set of all integral multiples of 4 under addition
$G_9 = \mathbb{R}^+$ under multiplication

4. Let A be a set and let c be one particular element of A.

Let $H = \{\mu \in S_A \mid \mu(c) = c\}$. Prove that H is a subgroup of S_A.

5. Compute the indicated product of cycles that are permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Write your answers in the form $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{pmatrix}$

(a) $(1, 5, 4, 3)(2, 3)(1, 2, 8, 7, 6)$

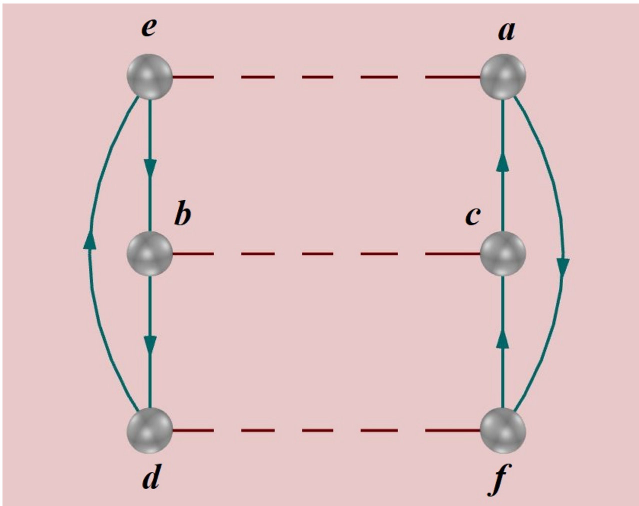
(b) $(2, 6)(1, 6, 4, 5, 7, 8)(7, 8)$

6. Express the permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles, and then as a product of transpositions.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 3 & 6 & 8 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 2 & 1 & 5 & 3 & 4 & 6 & 8 \end{pmatrix}$

7. Give the table for the group having the given digraph. Take e as the identity. List the identity e first in your table, and list the remaining elements alphabetically, so that your answers will be easy to check.



8. Find the number of elements in the indicated cyclic group.

(a) The cyclic subgroup of $\mathbb{Z}_{42}$ generated by 28.

(b) The cyclic subgroup of $\mathbb{Z}_{65}$ generated by 15.

9. Mark each of the following true or false. (No reasons required.)

(a) A binary operation on a set S may assign more than one element of S to some ordered pair of elements of S.

(b) Every finite group of at most three elements is abelian.

(c) Every set of numbers that is a group under addition is also a group under multiplication.

(d) Every group is isomorphic to some group of permutations.

(e) S_7 is isomorphic to the subgroup of all those elements of S_8 that leave the number 8 fixed.

(f) Every group of order ≤ 4 is cyclic.