

Quiz 1

1. (5 pts.) There is an isomorphism of U_9 with \mathbb{Z}_9 in which $\zeta = e^{i(2\pi/9)} \leftrightarrow 5$. Find the element in \mathbb{Z}_9 to which ζ^m must correspond for $m = 0, 2, 3, 4, 5, 6, 7$, and 8.

2. (5 pts.) Determine whether the binary operation $*$ defined is commutative and whether $*$ is associative.

$*$ defined on \mathbb{Z} by letting $a * b = ab - b^2$

3. (5 pts.) The map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = n + 3$ for $n \in \mathbb{Z}$ is one to one and onto \mathbb{Z} . Give the definition of a binary operation $*$ on \mathbb{Z} such that ϕ is an isomorphism mapping the first binary structure onto the second binary structure.

$\langle \mathbb{Z}, + \rangle$ onto $\langle \mathbb{Z}, * \rangle$

Find $7 * 9$. Find the identity element in $\langle \mathbb{Z}, * \rangle$.

4. (5 pts.) Determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ from the definition of group that does not hold.

Let $*$ be defined on the set \mathbb{R}^* of nonzero real numbers by letting $a * b = b/a$.

Quiz 2

Reminder: aH is a left coset; Ha is a right coset.

1. (4 pts.) Express $(1\ 3\ 7\ 5\ 6)(7\ 2\ 4\ 6)$ as a product of disjoint cycles.

2. (4 pts.) Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$ as a product of transpositions.

3. (4 pts.) Find the maximum possible order for an element of S_9 .

4. (4 pts.) Find all left cosets of the subgroup $\{\rho_0, \delta_2\}$ of the group D_4 given by the table on the supplemental page.

5. (4 pts.) Let G be a group, $H \leq G$, and $c, x \in H$. Prove $c \in Hx$ if and only if $xc^{-1} = h$ for some $h \in H$.

Quiz 3

1. (5 pts.) List, up to isomorphism, all abelian groups of order 225. How many different abelian groups of order 225 are there, up to isomorphism?

2. (5 pts.) What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_{15} \times \mathbb{Z}_{27}$? of $\mathbb{Z}_{20} \times \mathbb{Z}_{35}$?

3. (5 pts.) Prove that a direct product of abelian groups is abelian.

4. (5 pts.) The multiplication table for the group D_4 is given on the supplemental page. Here are the left cosets of the subgroup $\{\rho_0, \delta_2\}$.

$$\{\rho_0, \delta_2\}, \{\rho_1, \mu_1\}, \{\rho_2, \delta_1\}, \{\rho_3, \mu_2\}$$

Rewrite the multiplication table in the order exhibited by these left cosets. That is, use the order $\rho_0, \delta_2, \rho_1, \mu_1, \rho_2, \delta_1, \rho_3, \mu_2$. Do you seem to get a coset group of order 4? If so, is it isomorphic to \mathbb{Z}_4 or to the Klein 4-group V ?

Quiz 4

- (4 pts.) Find the characteristic of the given ring.
 - $\mathbb{Z}_{12} \times \mathbb{Z}_{10}$
 - $\mathbb{Z}_5 \times 2\mathbb{Z}$
- (4 pts.)
 - Give an example of an integral domain that is not a field.
 - Give an example of a commutative ring that does not have a unity.
- (4 pts.) Describe all units in the given ring.
 - \mathbb{Z}_{18}
 - $\mathbb{Z}_{10} \times \mathbb{Z}$
- (4 pts.) Find all solutions to the equation $r^2 + r - 12 = 0$ in \mathbb{Z}_{14} .
- (4 pts.) An element c of a ring R is called **idempotent** if $c^2 = c$. Show that a division ring contains exactly two idempotent elements.

Exam 1

- Let G be a group and let g be one fixed element of G . Show that the map ϕ_g , such that $\phi_g(x) = g^{-1}xg$ for $x \in G$, is an automorphism of G .
- Describe *all* group isomorphisms $\phi : \mathbb{Z}_{15} \rightarrow U_{15}$ by giving the possible values for $\phi(1)$.
- Find all subgroups of the group \mathbb{Z}_{28} , and draw the subgroup diagram for the subgroups.
- Give a careful proof that the following property of a binary structure $\langle S, * \rangle$ is indeed a structural property: For each $c \in S$, the equation $x * x = c$ has a solution x in S .
- Draw a Cayley digraph of the group \mathbb{Z}_{14} using the generating set $\{4, 7\}$.
- Consider the permutation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 3 & 6 \end{pmatrix}$ in S_6 . Find $|\langle \tau \rangle|$.
- Give a self-contained definition of a *group*.

Exam 2

- Express the permutation $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 5 & 7 & 8 & 1 & 3 & 4 \end{pmatrix}$ in S_8 as a product of disjoint cycles, and then as a product of transpositions. Is $\mu \in A_8$?
- Compute $\text{Ker } \phi$ and $\phi(8)$, where $\phi : \mathbb{Z}_{18} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_6$ is the group homomorphism satisfying $\phi(1) = (1, 4)$.
- Classify the group $(\mathbb{Z}_4 \times \mathbb{Z}_8) / \langle (2, 1) \rangle$ according to the fundamental theorem of finitely generated abelian groups.
- Give an example of a nontrivial homomorphism $\phi : \mathbb{Z}_{15} \rightarrow \mathbb{Z}_{12}$, if an example exists. If no such homomorphism exists, explain why that is so.
- List, up to isomorphism, all abelian groups of order 540. How many different abelian groups of order 540 are there, up to isomorphism?
- Let H and K be subgroups of a group G . Define \sim on G by $a \sim b$ if and only if $a = hbk$ for some $h \in H$ and some $k \in K$. Prove that \sim is an equivalence relation on G . Describe the elements in the equivalence class containing $a \in G$.
- Prove that \mathbb{Q}/\mathbb{Z} under addition is an infinite group in which every element has finite order.
- Let $\phi : G \rightarrow G'$ be a group homomorphism, and let N be a normal subgroup of G . Show that $\phi[N]$ is a normal subgroup of $\phi[G]$.