MTH 361
Exam 2
Spring 2010

100 points possible. Justify answers with supporting work.

1. (5 pts.) State **Lagrange's Theorem**.

2. (5 pts.) Let $\langle T, \triangle \rangle$ and $\langle S, \spadesuit \rangle$ be binary algebraic structures. Define **isomorphism** from T to S.

3. (5 pts.) Let N be a subgroup of a group H. Let $c \in H$. Define **left coset** of N containing c. (Use multiplicative notation.)

4. (5 pts.) Let $\langle F, \oplus \rangle$ and $\langle G, \heartsuit \rangle$ be groups. Define **homomorphism** from F to G.

5. (5 pts.) Define **kernel** of a homomorphism.

6. (5 pts.) Describe all automorphisms ψ of the group $\mathbb{Z}_{22}$ by giving the possible values for $\psi(1)$. How many automorphisms are there?

7. (5 pts.) Find all subgroups of the group $\mathbb{Z}_{24}$. (You don't have to write out all the elements of each of the subgroups, as long as you use correct notation for the subgroups.) How many subgroups are there?

8. (5 pts.) Find the maximum possible order for some element of

$$\mathbb{Z}_{12} \times \mathbb{Z}_{30} \times \mathbb{Z}_{36} \times \mathbb{Z}_{50}.$$

9. (5 pts.) Find the order of $(6, 8, 10)$ in $\mathbb{Z}_{20} \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

10. (5 pts.) List, up to isomorphism, all abelian groups of order 400.

11. (5 pts.) Find all left cosets of the subgroup $\langle(1,2)\rangle$ of the group $\mathbb{Z}_3 \times \mathbb{Z}_4$. How many distinct left cosets of $\langle(1,2)\rangle$ are there?

12. (5 pts.) Let $\phi : \mathbb{Z} \to S_9$ be the group homomorphism such that $\phi(1) = (1\ 5\ 6)(2\ 9\ 7\ 4)$. Find $\mathrm{Ker}(\phi)$ and $\phi(39)$.

13. (5 pts.) Give an example of a nontrivial homomorphism ϕ for the given groups, if an example exists. If no such example exists, briefly explain why that is so.

$$\phi : \mathbb{Z}_{15} \to \mathbb{Z}_8$$

14. (10 pts.) Let $\langle M_{2\times 2}(\mathbb{R}), +\rangle$ denote the additive group of 2×2 matrices with real entries, and let $\langle \mathbb{R}, +\rangle$ denote the additive group of real numbers. Define $\psi : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 3d & b \\ c & d - 2b \end{bmatrix},$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. Prove that ψ is a group homomorphism.

15. (15 pts.) Let ψ be a homomorphism from a group K to a group H. Prove that if N is a subgroup of H, then $\psi^{-1}[N]$ is a subgroup of K.

16. (10 pts.) Suppose G is a finite group with two or more elements. Suppose the only subgroups of G are $\{e\}$ and G. Prove that $|G|$ is a prime number