MTH 361
Exam 1
Spring 2010

100 points possible.

1. (10 pts.) Define **group**. (If you use technical terms in your statements of the group axioms, define those as well.)

2. (5 pts.) Let G be a group and let $y \in G$. Define the **cyclic subgroup of G generated by y.**

3. (5 pts.) Define what it means for two integers to be **congruent modulo** n.

4. (5 pts.) Define **permutation of a set**.

5. (5 pts.) Define **complete set of residues modulo** n.

6. (15 pts.) (a) Find the remainder when 2^{16} is divided by 7.

(b) Find the remainder when 2^{1355} is divided by 7.

7. (15 pts.) Determine whether the binary operation $*$ gives a group structure on the given set. If yes, just say so. If no, briefly state a reason.

(a) Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.

(b) Let $*$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by letting $a * b = a/b$.

(c) Let $*$ be matrix addition on the set

$$\mathbb{M} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbb{R} \right\}.$$

8. (15 pts.) In the group S_5, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}.$$

(a) Write σ in cycle notation.

(b) Let G be the cyclic subgroup generated by σ. Find the elements of G. (You may use permutation notation or cycle notation, your choice.)

(c) Let G be the cyclic subgroup generated by σ, as in part (b). Use a convenient symbol of your own choosing to denote each element of G that you found in part (b), then give the multiplication table for G. (Hint: You don't *really* have to carry out *all* those multiplications, do you?)

9. (15 pts.) Let H be a subgroup of a group G. Fix $g \in G$. Let

$$K = \{z \in G \mid \exists\ h \in H \text{ such that } z = ghg^{-1}\}.$$

Prove that K is a subgroup of G.

10. (10 pts.) Let H be a subgroup of a group G. Fix $g \in G$. Let

$$K = \{z \in G \mid \exists\ h \in H \text{ such that } z = ghg^{-1}\}.$$

Assume that K is a subgroup of G. Prove that if H is cyclic, then K is also cyclic.