Math 361
Exam 2
Spring 2006

100 points possible.

Important: For problems 7 through 11 (which are proofs), **do three of the five problems**. Clearly indicate which three you want me to grade (15 points each, 45 points total, no bonus credit).

Do **all** of problems 1 through 6.

0. (1 pt.) One free point for you today.

1. (9 pts.) Describe all group automorphisms $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{15}$ by giving the possible values for $\phi(1)$.

2. (9 pts.) Find the order of 18 in the cyclic group $\mathbb{Z}_{24}$.

3. (9 pts.) Find the order of $(1, 1)$ in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{12}$.

4. (9 pts.) Find all subgroups of $\mathbb{Z}_{18}$.

5. (9 pts.) List, up to isomorphism, all abelian groups of order 675.

6. (9 pts.) Let S_3 denote the group of all permuatations of $\{1, 2, 3\}$. In cycle notation,

$$S_3 = \{(1), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}.$$

Let $H = \langle (2\,3) \rangle$. Find all right cosets of H in S_3.

For problems 7 through 11, **do three of the five problems**. Clearly indicate which three you want me to grade. 15 points each, 45 points total, no bonus credit.

7. Define $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ as follows: $\forall (x, y), (z, w) \in \mathbb{Z} \times \mathbb{Z}$,

$$(x, y) \sim (z, w) \Longleftrightarrow x + w = y + z.$$

Prove that $\sim$ is an equivalence relation.

8. Let $+$ be defined on $\mathbb{Z} \times \mathbb{Z}$ in the usual way: $\forall (x, y), (z, w) \in \mathbb{Z} \times \mathbb{Z}$,

$$(x, y) + (z, w) = (x + z, y + w).$$

Define $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ as follows: $\forall (x, y), (z, w) \in \mathbb{Z} \times \mathbb{Z}$,

$$(x, y) \sim (z, w) \Longleftrightarrow x + w = y + z.$$

Assume that $\sim$ is an equivalence relation. Prove that $\sim$ is a congruence relation with respect to $+$.

9. Define $*$ on $\mathbb{Q}$ as follows: $\forall r, s \in \mathbb{Q}$,

$$r * s = 3rs - r - s + \frac{2}{3}.$$

Define $\phi : \mathbb{Q} \to \mathbb{Q}$ by $\phi(x) = 3x - 1$. Prove that ϕ is an isomorphism from $\langle \mathbb{Q}, * \rangle$ to $\langle \mathbb{Q}, \cdot \rangle$.

10. Let $\sim$ be an equivalence relation on a set S. Recall that $\overline{a} = \{x \in S \mid x \sim a\}$. Suppose $c, d \in S$ with $c \sim d$. Prove that $\overline{c} = \overline{d}$.

Prove it directly, from the definitions, not by invoking fancy theorems. Hints: How do you prove two sets are equal? How do you prove one set is a subset of another set? What does it mean, to say that $z \in \overline{a}$?

11. Let $\langle G, * \rangle$ be a group, and let $H \leq G$. Recall that $x * H = \{x * h \mid h \in H\}$. Suppose $a, b \in G$ and $a^{-1} * b \in H$. Prove that $b * H = a * H$.

Prove it directly, from the definitions, not by invoking fancy theorems. Hints: How do you prove two sets are equal? How do you prove one set is a subset of another set? What does it mean, to say that $z \in x * H$?