MTH 301
Exam 4
Spring 2013

100 points possible.

1. (6 pts.) Define **relation** from a set E to a set T.

2. (6 pts.) Let $f : A \to B$. Define what it means for f to be **one-to-one**.

3. (6 pts.) Let $f : A \to B$. Define what it means for f to be **onto**.

4. (6 pts.) Let R be a relation on a set A. Define what it means for R to be **transitive**.

5. (8 pts.) Let the universal set be the set $\mathbf{R}$ of all real numbers. Also let $A = \{x \in \mathbf{R} \mid 0 < x \leq 5\}$, $B = \{x \in \mathbf{R} \mid 2 \leq x < 8\}$, and $C = \{x \in \mathbf{R} \mid 6 \leq x < 10\}$. Find each of the following.

(a) $A \cap B$

(b) $A \cup C$

(c) $B^c \cap C^c$

6. (8 pts.) Let $V_i = \{x \in \mathbf{R} \mid 0 \leq x \leq i\} = [0, i]$ for all positive integers i.

(a) Find $\displaystyle\bigcup_{i=1}^{4} V_i$.

(b) Find $\displaystyle\bigcap_{i=1}^{n} V_i$.

(c) Find $\displaystyle\bigcup_{i=1}^{\infty} V_i$.

7. (8 pts.) Let S be the set of all strings of a's, b's, and c's.

Define $F : S \to \mathbf{Z}$ by letting $F(s) =$ the number of c's in s, for all $s \in S$.

(a) Is F one-to-one? Briefly explain your reasoning, or give a counterexample.

(b) Is F onto? Briefly explain your reasoning, or give a counterexample.

8. (8 pts.) Let $A = \{0, 1, 2, 3\}$ and let R be defined on A as follows.

$$R = \{(0, 1), (0, 3), (1, 0)\}$$

Find the transitive closure of R.

9. (8 pts.) Let $A = \{i \in \mathbf{Z} \mid -9 \leq i \leq 9\} = \{-9, -8, -7, \ldots, 7, 8, 9\}$. Let R be defined on A as follows.

$$\text{For all } x, y \in A, \quad x\ R\ y \quad \Leftrightarrow \quad 3 \mid (x - y)$$

It can be shown R is an equivalence relation on A (you don't have to prove it).

Find the distinct equivalence classes of R.

10. (12 pts.) Prove by mathematical induction.

$7^n - 1$ is divisible by 3, for each integer $n \geq 1$.

11. (12 pts.) Suppose that b_0, b_1, b_2, $\ldots$ is a sequence defined as follows:

$$\begin{aligned} &b_0 = 3,\ b_1 = 10, \\ &b_k = 7b_{k-1} - 12b_{k-2} \quad \text{for all integers } k \geq 2. \end{aligned}$$

Prove, using strong induction, that $b_n = 4^n + 2 \cdot 3^n$ for all integers $n \geq 0$.

12. (12 pts.) Here is a proof of a theorem. Fill in the blanks. (Note: Some blanks might be filled by a single variable, others by a short phrase, and others by one or more sentences.)

Theorem. *For any sets A, B, and C,* $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Proof. Suppose A, B, and C are particular but arbitrarily chosen sets.

Let $x \in A \cap (B \cup C)$. *[We must show that* $x \in$ ___(a)___*.]* By definition of intersection, $x \in$ ___(b)___ and $x \in$ ___(c)___. Thus, $x \in A$ and by definition of union, $x \in B$ or ___(d)___.

Case 1 ($x \in B$): In this case, since we know $x \in A$, then by definition of intersection $x \in$ ___(e)___, and so by definition of union, $x \in (A \cap B) \cup (A \cap C)$.

Case 2 ($x \notin B$): In this case, we must have $x \in C$. Since we know $x \in A$, then ___(f)___.

Hence in either case, $x \in (A \cap B) \cup (A \cap C)$ *[as was to be shown].*

[So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ *by definition of subset.]* ☐

(a)

(b)

(c)

(d)

(e)

(f)