Math 301
Exam 2
Spring 2008

Be neat and organized. Clearly indicate your answers. 100 points possible.

Part I — Computational Problems — 50 pts.

1. (10 pts.) Compute 17^{16} mod 35. Use an ordinary calculator for the arithmetic; support your answer with enough work to show you know what you're doing.

2. (10 pts.) Let $A = \{1, 2, 3\}$ and $B = \{3, 6, 9\}$.

Let $C = A \times \mathcal{P}(B)$. How many elements are in $\mathcal{P}(C)$?

3. (10 pts.) Let $\mathbf{Z}^{nonneg} = \{0, 1, 2, 3, \dots\}$. Let S be the set of all strings of a's, b's, and c's.

Define $F : S \to \mathbf{Z}^{nonneg}$ by letting $F(s) =$ the number of c's in s, for all $s \in S$.

Define $G : \mathbf{Z}^{nonneg} \to \mathbf{Z}^{nonneg}$ by letting $G(n) = n^2$, for all $n \in \mathbf{Z}^{nonneg}$.

Define $H : S \to \mathbf{Z}^{nonneg}$ by letting $H = G \circ F$.

(a) Find $H(aaba)$.

(b) Is H one-to-one? Briefly explain your reasoning, or give a counterexample.

(c) Is H onto? Briefly explain your reasoning, or give a counterexample.

4. (10 pts.) Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ defined by

$$R = \{(1, 1), (1, 5), (2, 2), (3, 3), (3, 4), (4, 3), (5, 1)\}.$$

Is R reflexive? symmetric? transitive? Briefly explain your reasoning, or give counterexamples.

5. (10 pts.) Eight points labeled A, B, C, D, E, F, G, H are arranged in a plane in such a way that no three lie on the same straight line. A triangle will be formed having three of the labeled points as vertices.

How many such triangles do not have B as a vertex?

Part II — Proofs — 50 pts.

Do three of these four problems.

6. ($16.\bar{6}$ pts.) Use regular mathematical induction to prove that $5^{2n} - 1$ is divisible by 24, for all integers $n \geq 1$.

7. ($16.\bar{6}$ pts.) Let $c_0 = 2$, $c_1 = 8$, and for integers $n > 1$ let $c_n = 8c_{n-1} - 15c_{n-2}$.

Prove, using strong induction, that $c_n = 3^n + 5^n$ for all integers $n \geq 0$.

8. ($16.\bar{6}$ pts.) Let D and E be sets. Prove by contradiction that if $E \subseteq D$, then $D^c \cap E = \varnothing$.

9. ($16.\bar{6}$ pts.) Let $f : X \to Y$, $A \subseteq Y$, and $B \subseteq Y$.

Prove $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.