Math 301
Exam 2
Fall 2007

100 points possible.

1. (4 pts.) For the given equivalence relation, find the requested equivalence class.

$R = \{(1,1),(1,3),(2,2),(3,1),(3,3),(4,4)\}$ on $\{1,2,3,4\}$. Find $[1]$.

2. (8 pts.) How many different equivalence relations are possible on a three-element set?

3. (8 pts.) Let $A = \{2,3,4\}$ and $B = \{7,8\}$. Write down all functions $f : A \to B$, using any convenient shorthand notation. Indicate which are one-to-one and which are onto B.

4. (16 pts.)

(a) How many different anagrams (including nonsensical words) can be made from ASHLEYTISDALE given that the first letter must remain A and the last letter must remain E.

(b) How many different anagrams (including nonsensical words) can be made from JUSTINTIMBERLAKE given that the J must remain before the K (but not necessariliy contiguous with each other).

5. (16 pts.) A standard deck of 52 cards contains 4 different suits, with 13 different ranks in each suit.

(a) How many 8-card hands are there with all 8 cards of the same suit?

(b) How many 8-card hands are there with exactly two tens, and no other repeated ranks?

6. (8 pts.) Write the contrapositive of each statement. (Do not prove!)

(a) If $x \; R \; y$, then $x \neq y$.

(b) If $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

7. (8 pts.) Let $\mathcal{K}$ be a set of undefined elements, and let R be an undefined relation on K. Assume the following postulates.

POSTULATE 0. $\forall x, y \in \mathcal{K}$, $x \; R \; y$ or $x \not R \; y$ but not both.

POSTULATE 1. $\forall x, y \in \mathcal{K}$, if $x \neq y$, then either $x \; R \; y$ or $y \; R \; x$.

POSTULATE 2. $\forall x, y \in \mathcal{K}$, if $x \; R \; y$, then $x \neq y$.

POSTULATE 3. $\forall x, y, z \in \mathcal{K}$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

POSTULATE 4. $\mathcal{K}$ consists of exactly four distinct elements.

Here are proofs of Theorems 1 and 2. Fill in the blanks.

THEOREM 1. $\forall a, b \in \mathcal{K}$, if $a \; R \; b$, then $b \not R \; a$.

Proof. Let $a, b \in \mathcal{K}$. Suppose both $a \; R \; b$ and $b \; R \; a$. Then by ____________, we have $a \; R \; a$. But this is impossible by ____________. $\otimes$

THEOREM 2. $\forall a, b, c \in \mathcal{K}$, if $a \; R \; b$, then $a \; R \; c$ or $c \; R \; b$.

Proof. Let $a, b, c \in \mathcal{K}$. Either $c = a$ or $c \neq a$.

Case 1: $c = a$. Then $c \; R \; b$, and we are done.

Case 2: $c \neq a$. Then by ____________, we have $a \; R \; c$ or $c \; R \; a$. If $a \; R \; c$, we are done. If $c \; R \; a$, then since we also have ____________, we conclude by ____________ that $c \; R \; b$. $\square$

8. (16 pts.) Use regular mathematical induction to prove for all positive integers n,

$$5 + 11 + 17 + \cdots + (6n - 1) = 3n^2 + 2n.$$

9. (16 pts.) Given: $b_0 = 1$, $b_1 = 10$, and for $n \geq 2$, $b_n = 10b_{n-1} - 25b_{n-2}$.

Use strong mathematical induction to prove $b_n = 5^n + n5^n$ for all integers $n \geq 0$.